Quantum spin ladder systems associated with *su*(2|2)

LETTER TO THE EDITOR

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Abstract
Two integrable quantum spin ladder systems will be introduced associated with the fundamental $su(2|2)$ solution of the Yang–Baxter equation. The first model is a generalized quantum Ising system with Ising rung interactions. In the second model the addition of extra interactions allows us to impose Heisenberg rung interactions without violating integrability. The existence of a Bethe ansatz solution for both models allows us to investigate the elementary excitations for antiferromagnetic rung couplings. We find that the first model does not show a gap whilst in the second case there is a gap for all positive values of the rung coupling.

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The study of spin ladder systems continues to receive a great deal of interest from both theoretical and experimental points of view providing new insights into low-dimensional quantum systems. By now it is well established that the Heisenberg spin-$\frac{1}{2}$ ladders with an even number of legs are found to have a finite energy gap while those with an odd number of legs show gapless spin excitations and these results been verified experimentally in a number of systems [1]. Under hole-doping these ladders are predicted to pair and may superconduct. The quantum phase transition between the gap and gapless phase also proves to be both theoretically and experimentally interesting. Several generalized ladders incorporating interactions beyond simple rung and leg exchange have appeared in the literature and demonstrate remarkably interesting behaviour in relation to the ground state structure and excitation spectra (see, for example, [2, 3]).

Another approach has been to derive quasi two-dimensional systems using the well established theories from one dimension. A number of integrable ladder systems involving tunable interaction parameters can been found in the literature for which some thermodynamic

quantities have been obtained [4–15]. An example is the two-leg zigzag Majumdar–Ghosh ladder [4], which was generalized in [5] using higher-order conservation laws of the Heisenberg chain to obtain an exactly solvable model. Along the same lines, Frahm and Rödenbeck used an algebra homomorphism to develop an integrable family of coupled Heisenberg spin-$\frac{1}{2}$ chains [6].

Various ladder models have been developed by an extension of the symmetry algebra [8, 9, 11–13, 16]. Families of n-leg spin-$\frac{1}{2}$ ladders were obtained from $su(N)$ spin chains in [11] and this work was extended in [17]. An $su(2)$-invariant spin-$\frac{1}{2}$ ladder model was introduced in [8] and showed both isotropic exchange and biquadratic interactions. A solution of this model was obtained in [12] by showing that the model can be made to satisfy the Hecke algebra. Another example is the ladder model formulated using the Hubbard model in [15]. A new set of novel spin ladder models possessing extra free parameters were introduced in [14], which were shown to reduce to those of Wang [16] for the $su(4)$ and $su(3|1)$.

In this work we present two new solvable ladder systems with coupled rung interactions. We begin with the $su(2|2)$ invariant solution of the Yang–Baxter equation [18] which under a Jordan–Wigner transformation can be considered as a generalized quantized Ising model. Ising rung interactions can be coupled into the model without corrupting the integrability of the system. However, an attempt to directly introduce Heisenberg interactions fails. To overcome this, we apply a basis transformation, which yields a Hamiltonian which does commutes with the Heisenberg rung interaction. We present the Bethe ansatz solution and the resulting expression for the energy is obtained. An analysis of the Bethe ansatz solution shows that for the generalized Ising case there is no gap for the elementary excitations. On the other hand, the model with Heisenberg interactions does show a gap for all positive values of the coupling parameter J. This situation is in some contrast to other recently proposed generalized spin ladder systems formulated within the quantum inverse scattering method (QISM) [13, 15, 16]. In these cases the gap closes at some critical value greater than zero. Numerical and experimental results suggest that the gap should prevail for all positive values of J [1].

The first model has a local Hamiltonian of the following form:

$$H_{i,j} = \left[\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+ + \tfrac{1}{2}(\sigma_i^z + \sigma_j^z) \right] \left[\tau_i^+ \tau_j^- + \tau_i^- \tau_j^+ + \tfrac{1}{2}(\tau_i^z + \tau_j^z) \right] + \tfrac{1}{2} J (\sigma_i^z \tau_i^z + \sigma_j^z \tau_j^z) \quad (1)$$

where J above can take arbitrary real values. The sets $\{\sigma_j^\pm, \sigma_j^z\}$ and $\{\tau_j^\pm, \tau_j^z\}$ are two commuting sets of Pauli matrices acting on site j which describe interactions on the two legs of the ladder.

The global Hamiltonian

$$H = \sum_{j=1}^{N} H_{j,j+1}$$

with periodic boundary conditions is integrable as a result of the QISM. In this case, the R-matrix satisfying the Yang–Baxter equation is the fundamental $su(2|2)$ solution [18] from which the Hamiltonian is derived in the usual manner through the QISM for $J = 0$. It is pertinent to mention at this point that the R-matrix solution of the Yang–Baxter equation is used in an ungraded context here. This is achieved by means of the well known Jordan–Wigner transformation mapping between canonical spin and Fermi operators. We can arbitrarily couple (with coupling J) the Ising rung interactions in the model above by virtue of the fact that these terms commute with the Hamiltonian (1).

In order to obtain a model with Heisenberg rung interactions we introduce additional terms

into the Hamiltonian which now reads as

$$\check{H} = \sum_{j=1}^{N} \left\{ \left[\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \tfrac{1}{2}(\sigma_j^z + \sigma_{j+1}^z) \right] \left[\tau_j^+ \tau_{j+1}^- + \tau_j^- \tau_{j+1}^+ + \tfrac{1}{2}(\tau_j^z + \tau_{j+1}^z) \right] \right.$$

$$- (\sigma_j^+ \tau_j^- + \sigma_j^- \tau_j^+)(\sigma_{j+1}^+ \tau_{j+1}^- + \sigma_{j+1}^- \tau_{j+1}^+) - \tfrac{1}{4}(\sigma_j^z - \tau_j^z)(\tau_{j+1}^z - \sigma_{j+1}^z)$$

$$\left. + \tfrac{1}{2} J (\boldsymbol{\sigma}_j \cdot \boldsymbol{\tau}_j + \boldsymbol{\sigma}_{j+1} \cdot \boldsymbol{\tau}_{j+1}) \right\}. \tag{2}$$

In the absence of rung interactions, the local Hamiltonians for the two models are related through the following basis transformation:

$$\begin{aligned}
| \uparrow, \uparrow \rangle &\rightarrow | \uparrow, \uparrow \rangle \\
| \uparrow, \downarrow \rangle &\rightarrow \tfrac{1}{\sqrt{2}} (| \uparrow, \downarrow \rangle - | \downarrow, \uparrow \rangle) \\
| \downarrow, \uparrow \rangle &\rightarrow \tfrac{1}{\sqrt{2}} (| \uparrow, \downarrow \rangle + | \downarrow, \uparrow \rangle) \\
| \downarrow, \downarrow \rangle &\rightarrow | \downarrow, \downarrow \rangle.
\end{aligned} \tag{3}$$

Note that it is possible to include more general interactions in both models, such as an applied magnetic field, which will commute with these Hamiltonians. However, here we will limit ourselves to only those described above.

It can been seen for both models that besides the rung interaction, there are Ising interactions along the legs and in addition there exist *XX* correlated exchanges and four-body biquadratic *XX* exchange interactions. In the absence of rung interactions both models retain non-trivial spin interactions between the legs.

From the Bethe ansatz solution which we present below, we can determine the behaviour of the elementary excitations for the systems. For the Hamiltonian (1), the energy levels are given by

$$E = \sum_{j=1}^{n_1+n_2+n_3} \frac{1}{\lambda_j^2 + \frac{1}{4}} - N - J(N - 2n_2 - 2n_3) \tag{4}$$

while for the second model (2) they read as

$$\check{E} = \sum_{j=1}^{n_1+n_2+n_3} \frac{1}{\lambda_j^2 + \frac{1}{4}} - N - J[3N - 4(n_1 + n_2 + n_3)]. \tag{5}$$

Here N is the lattice length and n_1, n_2, n_3 are quantum numbers for the eigenstates. In both cases the variables λ_j are solutions of the following Bethe ansatz equations (BAE). These equations are different from those given, for example, in [19] since the Jordan–Wigner transformation from Fermi operators to spin operators induces sector-dependent twisted boundary conditions. However, the procedure to obtain these equations in this case follows in much the same way so we simply give the final result:

$$\left(\frac{\lambda_k - i/2}{\lambda_k + i/2} \right)^N = \prod_{j=1}^{n_2+n_3} \frac{\lambda_k - \mu_j + i/2}{\lambda_k - \mu_j - i/2} \prod_{l=1, l \neq k}^{n_1+n_2+n_3} \frac{\lambda_l - \lambda_k + i}{\lambda_l - \lambda_k - i} \qquad k = 1, \ldots, n_1 + n_2 + n_3$$

$$\prod_{m=1}^{n_3} \frac{\nu_m - \mu_j + i/2}{\nu_m - \mu_j - i/2} = -(-1)^{(N+n_2+n_3)} \prod_{k=1}^{n_1+n_2+n_3} \frac{\lambda_k - \mu_j + i/2}{\lambda_k - \mu_j - i/2} \qquad j = 1, \ldots, n_2 + n_3 \qquad (6)$$

$$\prod_{l=1, l \neq m}^{n_3} \frac{\nu_l - \nu_m + i}{\nu_l - \nu_m - i} = \prod_{j=1}^{n_2+n_3} \frac{\nu_m - \mu_j - i/2}{\nu_m - \mu_j + i/2} \qquad m = 1, \ldots, n_3.$$

For the first model (1), there is no gap when $J = 0$. For large values of J, in which case the rung interactions dominate, there is still no gap due to the non-uniqueness of the ground

state for a two-site Ising system. This observation is confirmed from the BAE. An elementary excitation is characterized by the quantum numbers $n_2 = n_3 = 0$ and $n_1 = 1$. Solving the BAE yields the energy expression from (4)

$$E = \frac{1}{\lambda^2 + \frac{1}{4}} - N - JN \tag{7}$$

where $\lambda = \frac{1}{2} \cot(\pi k/N)$, $k = 1, \ldots, N - 1$. Choosing $k = 1$, then the thermodynamic limit $N \to \infty$ shows that $\lambda \to \infty$ and hence there is no gap.

Now we turn our attention to the second model (2). Again there is no gap for $J = 0$. In the limit of large J, the ground state consists of a product of rung singlets indicating a gap to any excited state. Using the same solution of the BAE as in the above case, the energy of an excited state in this model is given from (5) by

$$\check{E} = \frac{1}{\lambda^2 + \frac{1}{4}} - N - J(3N - 4). \tag{8}$$

It is apparent that there is a gap of

$$\Delta = \frac{1}{\lambda^2 + \frac{1}{4}} + 4J. \tag{9}$$

In the thermodynamic limit we can allow $\lambda \to \infty$ which shows that a gap of $\Delta = 4J$ persists. Our analysis shows that there is always a gap for this model in the case of antiferromagnetic Heisenberg interactions along the rungs of the ladder. Only in the limit of vanishing coupling J does the gap close.

In conclusion, we make the observation that the behaviour of the models (1) and (2) is in some constrast to the usual two-leg Heisenberg ladder. We find in these models that the ground state consists of a product of rung states for all values of the coupling $J > 0$ while in the Heisenberg ladder this behaviour is true in the strong-coupling limit but, for weak coupling, strong quantum fluctuations arise. We emphasize that this character in models (1) and (2) is not dependent on the overall sign of the bulk Hamiltonian. In each case the energy spectrum (when $J = 0$) is symmetric about the zero energy level.

References

[1] Dagotto E and Rice T M 1996 *Science* **271** 618
    (Dagotto E and Rice T M 1995 *Preprint* cond-mat/9509181)
[2] Nersesyan A A and Tsvelik A M 1997 *Phys. Rev. Lett.* **78** 3939
[3] Kolezhuk A K and Mikeska H J 1998 *Phys. Rev. Lett.* **80** 2709
    Kolezhuk A K and Mikeska H J 1998 *Int. J. Mod. Phys.* **B12** 2325
    Kolezhuk A K and Mikeska H J 1997 *Phys. Rev.* B **56** R11
[4] Majumdar C K and Ghosh D K 1969 *J. Math. Phys.* **10** 1399
[5] Muramoto N and Takahashi M 1999 *J. Phys. Soc. Japan* **68** 2098
    (Muramoto N and Takahashi M 1999 *Preprint* cond-mat/9902007)
[6] Frahm H and Rödenbeck C 1996 *Europhys. Lett.* **33** 47
    Frahm H and Rödenbeck C 1999 *Eur. Phys. J.* B **10** 409
    (Frahm H and Rödenbeck C 1998 *Preprint* cond-mat/9812103)
    Frahm H and Rödenbeck C 1997 *J. Phys. A: Math. Gen.* **30** 4467
    (Frahm H and Rödenbeck C 1997 *Preprint* cond-mat/9702083)

[7] Popkov V Yu and Zvyagin A A 1993 *Phys. Lett.* A **175** 295
    Zvyagin A A 1995 *Phys. Rev.* B **51** 12579
[8] Albeverio S, Fei S-M and Wang Y 1999 *Europhys. Lett.* **47** 364
    (Albeverio S, Fei S-M and Wang Y 1998 *Preprint* cond-mat/9807341)
[9] Frahm H and Kundu A 1999 *J. Phys. C: Condens. Matter* **11** L557
    (Frahm H and Kundu A 1999 *Preprint* cond-mat/9910104)
[10] Scalapino D, Zhang S-C and Hanke W 1998 *Phys Rev.* B **58** 443
[11] Batchelor M T and Maslen M 1999 *J. Phys. A: Math. Gen.* **32** L377
    (Batchelor M T and Maslen M 1999 *Preprint* cond-mat/9907134)
[12] Batchelor M T and Maslen M 2000 *J. Phys. A: Math. Gen.* **33** 443
    (Batchelor M T and Maslen M 1999 *Preprint* cond-mat/9907480)
[13] de Gier J, Batchelor M T and Maslen M 2000 *Phys. Rev.* B **61** 15196
    (de Gier J, Batchelor M T and Maslen M 1999 *Preprint* cond-mat/9911074)
[14] Foerster A, Links J and Prestes Tonel A 2000 Integrable generalised spin ladder models *Preprint* cond-mat/0005463
[15] Links J and Foerster A 2000 *Phys Rev.* B **62** 65
    (Links J and Foerster A 1999 *Preprint* cond-mat/9911096)
[16] Wang Y 1999 *Phys. Rev.* B **60** 9236
    (Wang Y 1999 *Preprint* cond-mat/9901168)
[17] Batchelor M T, de Gier J, Links J and Maslen M 2000 *J. Phys. A: Math. Gen.* **A33** L97
    (Batchelor M T, de Gier J, Links J and Maslen M 1999 *Preprint* cond-mat/9911043)
[18] Essler F H L, Korepin V E and Schoutens K 1992 *Phys. Rev. Lett.* **68** 2960
[19] Essler F H L, Korepin V E and Schoutens K 1994 *Int. J. Mod. Phys.* B **8** 3205